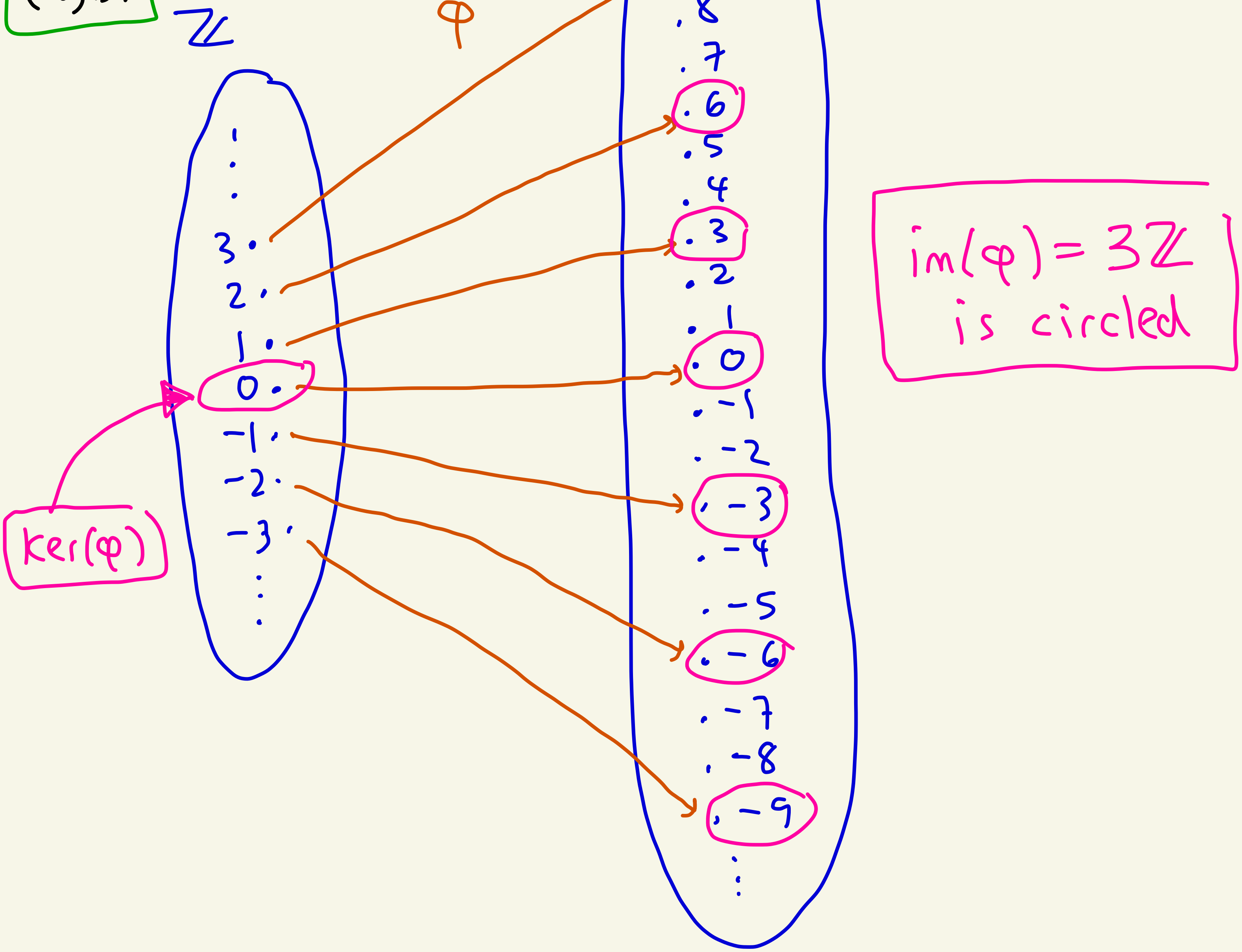
Math 4550
Homework 4
Solutions

1

(a,b)



$\ker(\varphi) = \{0\}$

$\operatorname{im}(\varphi) = 3\mathbb{Z}$

(c)

Given $m, n \in \mathbb{Z}$ we have

$$\varphi(m+n) = 3(m+n) = 3m + 3n = \varphi(m) + \varphi(n)$$

$\boxed{\text{(d)}}$

φ is one-to-one:

Suppose $\varphi(m) = \varphi(n)$ for some $m, n \in \mathbb{Z}$.

Then $3m = 3n$.

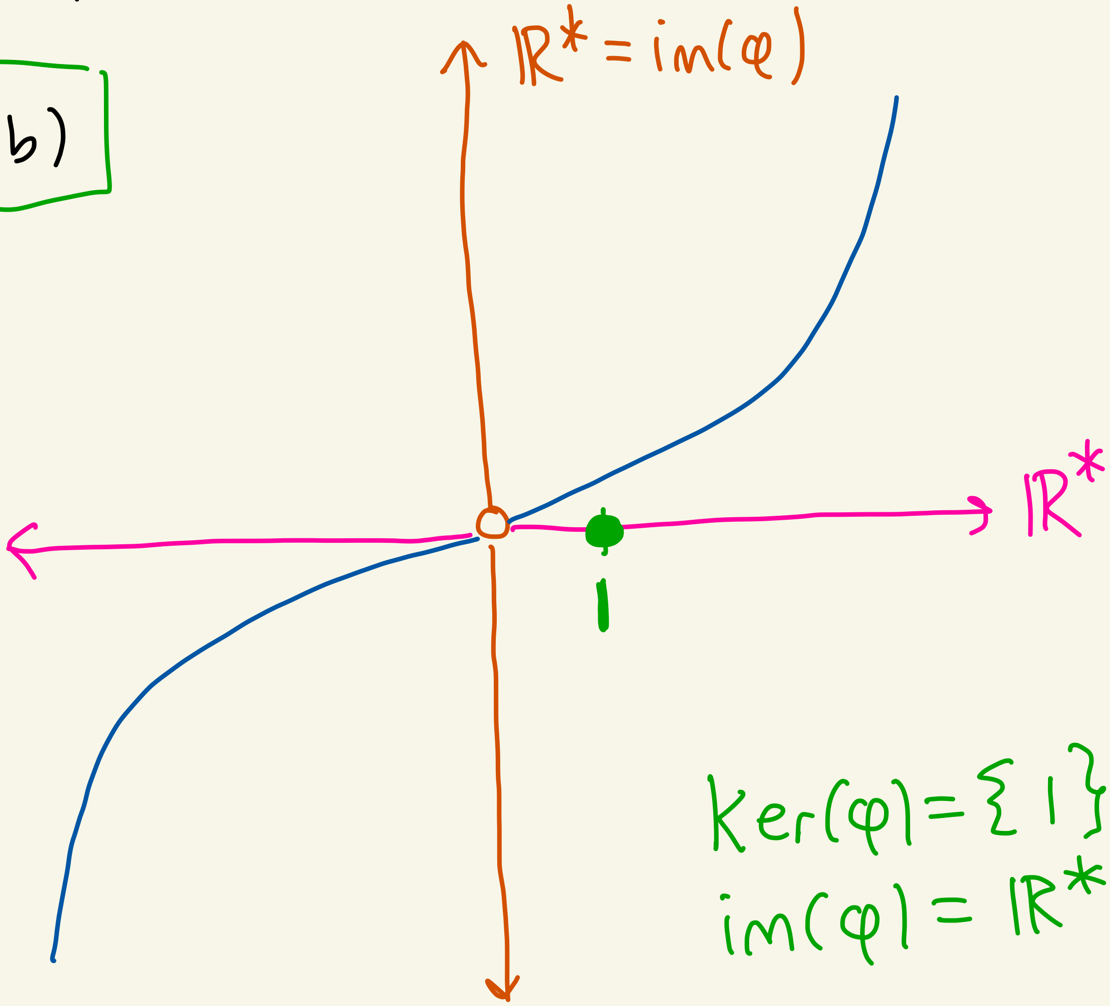
So, $m = n$

φ is not onto:

$2 \in \mathbb{Z}$ but there is no $n \in \mathbb{Z}$ with $\varphi(n) = 2$ since that would require $3m = 2$ or $m = \frac{2}{3}$ which is not in $\mathbb{Z}$.

② Recall that $\mathbb{R}^* = \mathbb{R} - \{0\}$ is a group under multiplication.

$\boxed{(a,b)}$



$\ker(\varphi) = \{1\}$
$\text{im}(\varphi) = \mathbb{R}^*$

The identity of $\mathbb{R}^*$ is 1.

So, $\ker(\varphi) = \{x \mid \varphi(x) = 1\}$

$= \{x \mid x^3 = 1\}$

$= \{1\}$

$\boxed{(c)}$

φ is a homomorphism:

Given $x, y \in \mathbb{R}^*$ we have that

$$\varphi(xy) = (xy)^3 = x^3 y^3 = \varphi(x)\varphi(y)$$

φ is one-to-one:

Suppose $\varphi(x) = \varphi(y)$ for some $x, y \in \mathbb{R}^*$.

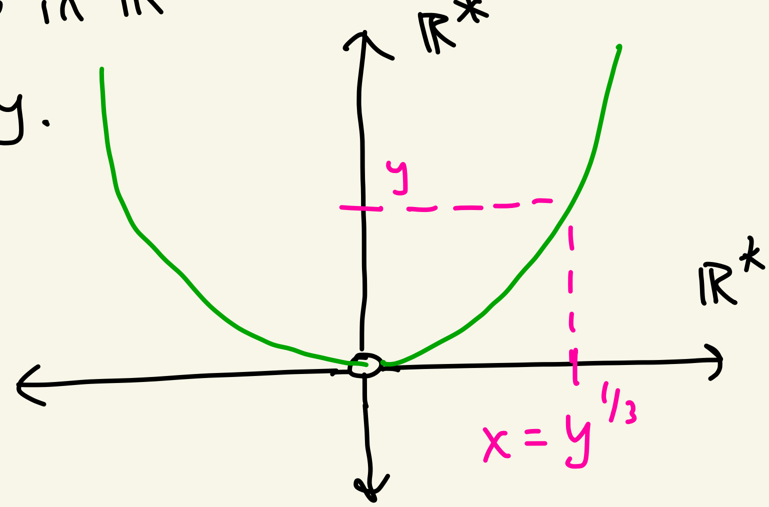
$$\left.\begin{aligned} &\text{Then, } x^3 = y^3. \\ &\text{So, } (x^3)^{1/3} = (y^3)^{1/3}. \\ &\text{Thus, } x = y. \end{aligned}\right\} \text{using properties of cube root}$$

φ is onto:

Let $y \in \mathbb{R}^*$

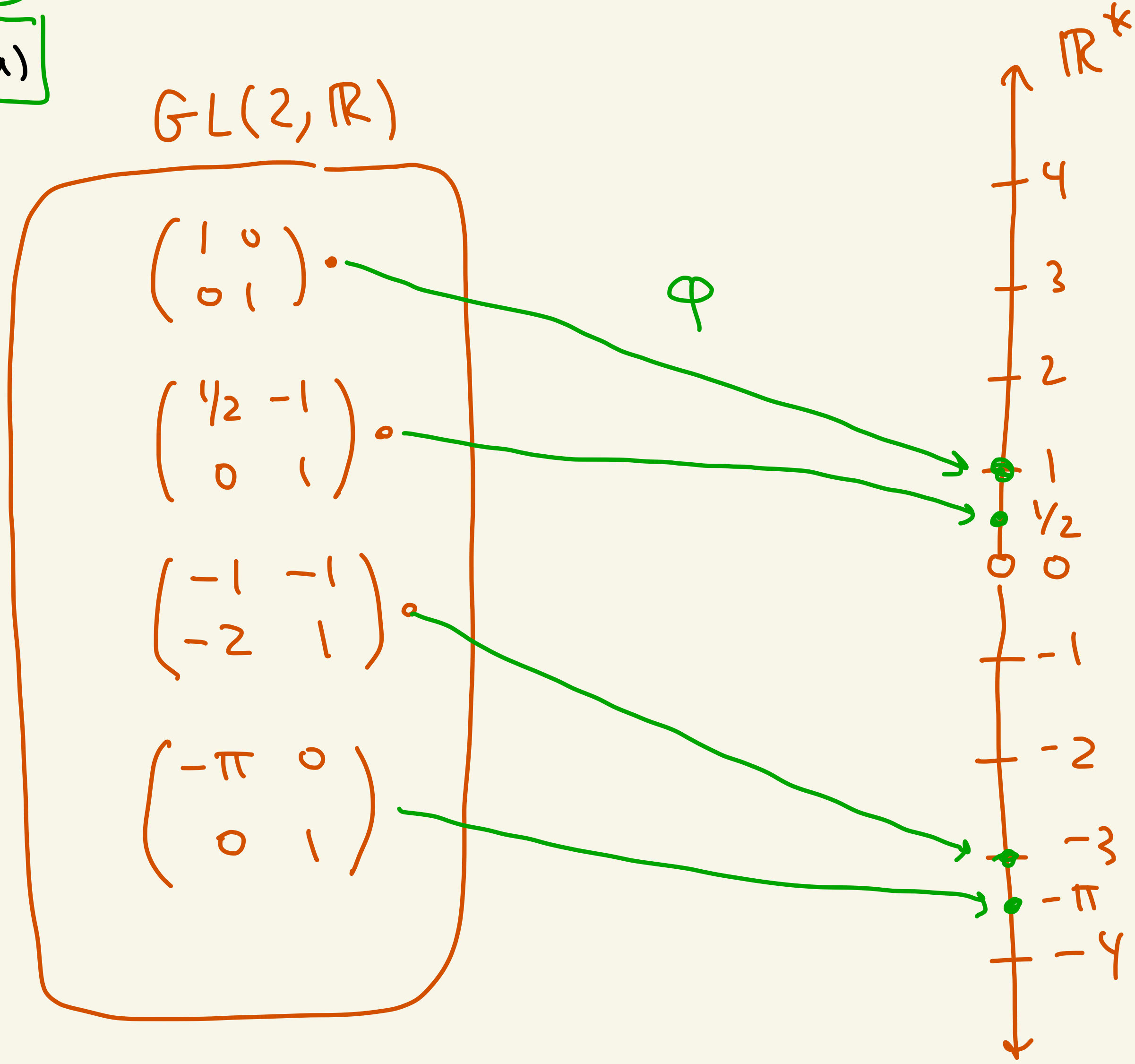
Set $x = y^{1/3}$ which is also in $\mathbb{R}^*$

Then, $\varphi(x) = x^3 = (y^{1/3})^3 = y$.



3

(a)



(b) Given $A, B \in GL(2, \mathbb{R})$ we have

$$\varphi(AB) = \det(AB) = \det(A)\det(B) = \varphi(A)\varphi(B)$$

Property of determinants

(c)

<u>φ is onto:</u>

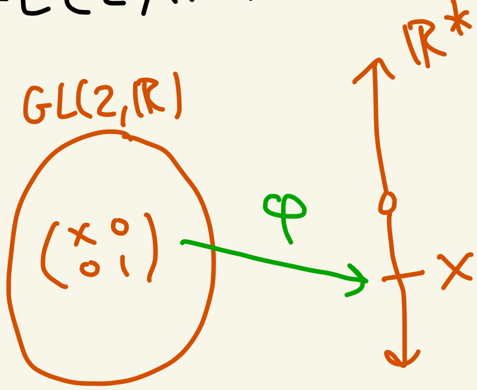
Given $x \in \mathbb{R}^*$ set $A = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$.

Since $x \in \mathbb{R}^*$ we know $x \neq 0$, thus

$\det(A) = x \neq 0$ so $A \in GL(2,\mathbb{R})$.

And $\varphi(A) = \det(A) = x$.

Thus, φ is onto $\mathbb{R}^*$.



<u>φ is not one-to-one:</u>

$\varphi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

$\varphi\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} = \det\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} = 1$

So, $\varphi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \varphi\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$

Thus φ is not one-to-one.

$\boxed{\text{(d)}}$ Recall that 1 is the identity of $\mathbb{R}^*$.

$$\ker(\varphi) = \{ A \mid A \in GL(2,\mathbb{R}) \text{ where } \varphi(A) = 1 \}$$
$$= \{ A \mid A \in GL(2,\mathbb{R}) \text{ where } \det(A) = 1 \}$$
$$= SL(2,\mathbb{R})$$

From (c) we saw that φ is onto.

Thus, $\operatorname{im}(\varphi) = \mathbb{R}^*$.

$\blacksquare$

4

(a) $\varphi(4) = \varphi(1+1+1+1) \overset{\varphi \text{ is a homomorphism}}{=} \varphi(1) + \varphi(1) + \varphi(1) + \varphi(1)$

$\overset{\varphi(1)=5 \text{ is given}}{=} 5+5+5+5 = \boxed{20}$

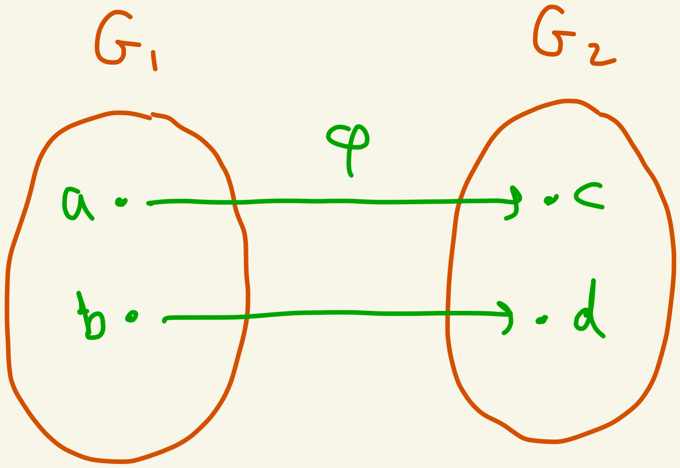
(b) Since $\varphi(1) = 5$
We know $\varphi(-1) = -5$.

theorem from class:
$\varphi(x^{-1}) = [\varphi(x)]^{-1}$

(c)
$\varphi(-3) = \varphi((-1)+(-1)+(-1))$
$\overset{\varphi \text{ is a homomorphism}}{=} \varphi(-1) + \varphi(-1) + \varphi(-1)$
$\overset{\text{part (b)}}{=} -5-5-5 = \boxed{-15}$

5

(a) Let $c, d \in G_2$.



Since φ is onto there exists $a, b \in G_1$ with $\varphi(a) = c$ and $\varphi(b) = d$.

Then,

$$cd = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = dc$$

φ is a homomorphism

G_1 is abelian so $ab = ba$

φ is a homomorphism

Thus, $cd = dc$.

So, G_2 is abelian

(b)

$\mathbb{Z}_6$ is abelian

D_6 is not abelian

By part (a) they cannot be isomorphic.

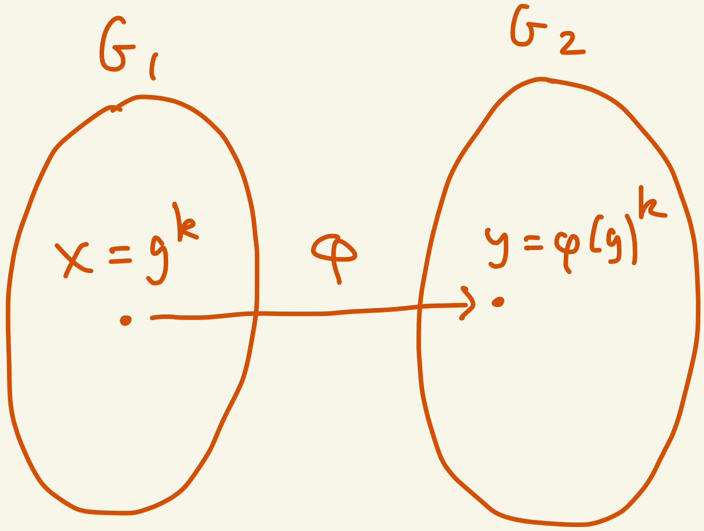
$\boxed{(c)}$

Since G_1 is cyclic there exists $g \in G_1$ with $G_1 = \langle g \rangle$.

Let's show that $G_2 = \langle \varphi(g) \rangle$.

Let $y \in G_2$.

Since φ is onto there exists $x \in G_1$ with $\varphi(x) = y$.



Since $G_1 = \langle g \rangle$ we have that $x = g^k$ for some $k \in \mathbb{Z}$.

Then,

$$y = \varphi(x) = \varphi(g^k) = [\varphi(g)]^k$$

$\boxed{\varphi \text{ is a homomorphism}}$

Thus, $\varphi(g)$ generates G_2.

$\boxed{(d)}$ $\mathbb{Z}_4$ is cyclic

$\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic

Thus by (c) they are not isomorphic

6

(a) We have $\varphi(e_1) = \varphi(e_1 e_1) \overset{\varphi \text{ is a homomorphism}}{=} \varphi(e_1)\varphi(e_1)$

So, $\varphi(e_1)^{-1}\varphi(e_1) = \varphi(e_1)^{-1}\varphi(e_1)\varphi(e_1)$

Thus, $e_2 = e_2\varphi(e_1)$

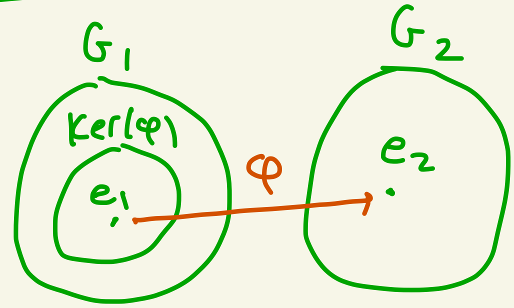
So, $e_2 = \varphi(e_1)$ $\boxed{//}$

(b) We have that

$$\varphi(x)\varphi(x^{-1}) \overset{\varphi \text{ is a homomorphism}}{=} \varphi(xx^{-1}) = \varphi(e_1) \underset{\text{part (a)}}{=} e_2$$

Thus, $[\varphi(x)]^{-1} = \varphi(x^{-1})$ $\boxed{//}$

(c) $\operatorname{Ker}(\varphi) = \{x \in G_1 \mid \varphi(x) = e_2\}$



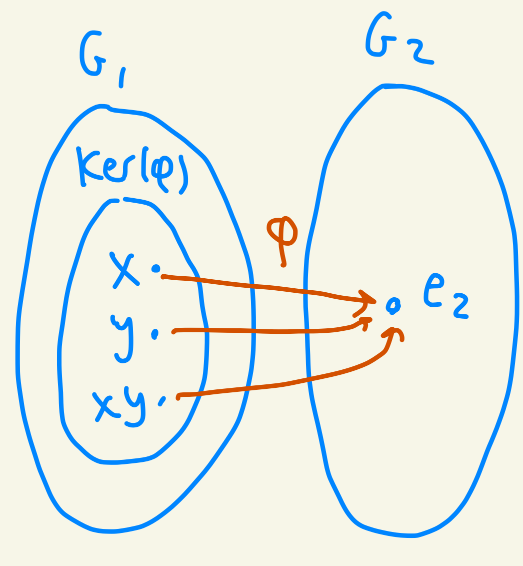
(i) By (a) we know $\varphi(e_1) = e_2$

So, $e_1 \in \operatorname{Ker}(\varphi)$.

(ii) Given $x, y \in \operatorname{Ker}(\varphi)$ we have

$$\varphi(xy) \underset{\substack{\varphi \text{ is a} \\ \text{homomorphism}}}{=} \varphi(x)\varphi(y) \underset{\substack{\text{since} \\ x,y \in \operatorname{Ker}(\varphi)}}{=} e_2 e_2 = e_2$$

So $xy \in \ker(\varphi)$.



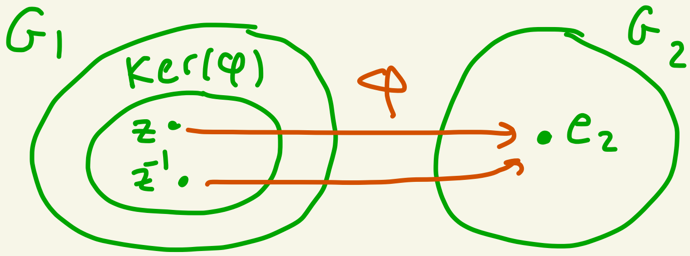
(iii) Let $z \in \ker(\varphi)$.

Then, $\varphi(z) = e_2$.

So, $[\varphi(z)]^{-1} = e_2^{-1}$

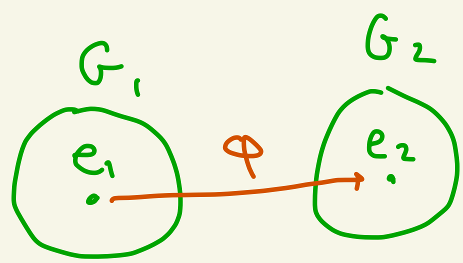
Thus, $\varphi(z^{-1}) = e_2$

Thus, $z^{-1} \in \ker(\varphi)$.



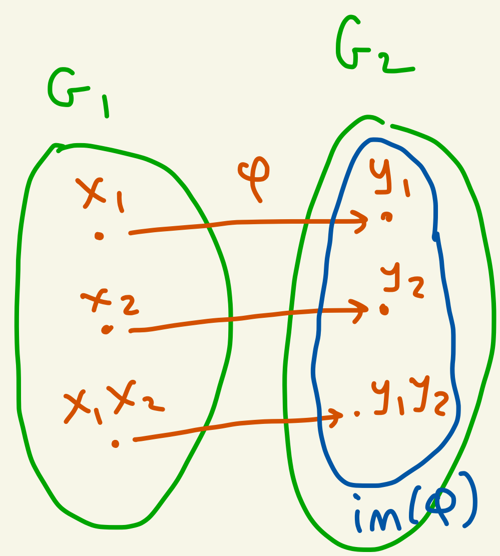
By (i)–(iii) we know $\ker(\varphi) \leq G_1$. $\boxed{//}$

(d) $\operatorname{im}(\varphi) = \{\varphi(x) \mid x \in G_1\}$.



(i) We know $e_2 = \varphi(e_1)$.
So, $e_2 \in \operatorname{im}(\varphi)$.

(ii) Let $y_1, y_2 \in \operatorname{im}(\varphi)$.
Then there exist
$x_1, x_2 \in G_1$ with
$\varphi(x_1) = y_1$ and $\varphi(x_2) = y_2$

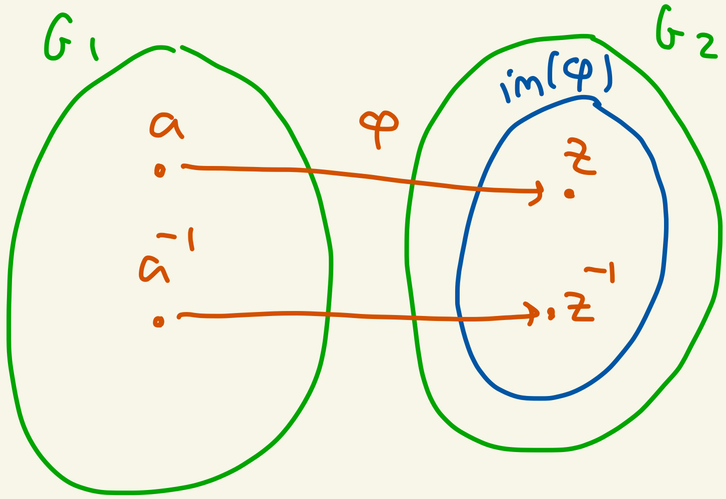


Thus,
$$\varphi(x_1x_2) = \varphi(x_1)\varphi(x_2) = y_1y_2.$$
Since $x_1x_2 \in G$ this shows
that $y_1y_2 \in \operatorname{im}(\varphi)$.

(iii) Let $z \in \operatorname{im}(\varphi)$.
Then there exists
$a \in G_1$ with $\varphi(a) = z$.

Thus,

$$\varphi(a^{-1}) \underset{\text{part (b)}}{=} [\varphi(a)]^{-1} = z^{-1}$$



Since $a^{-1} \in G_1$ we know $z^{-1} \in \text{im}(\varphi)$

By (i)-(iii) we have that

$\text{im}(\varphi) \leq G_2$.

$\boxed{\text{(e)}}$

($\Rightarrow$) Suppose φ is one-to-one.

By part (a) we know $\varphi(e_1) = e_2$

and thus $e_1 \in \ker(\varphi)$.

So, $\{e_1\} \subseteq \ker(\varphi)$.

Let $x \in \ker(\varphi)$.

Then $\varphi(x) = e_2$

Since $\varphi(x) = e_2 = \varphi(e_1)$ and φ is one-to-one this gives $x = e_1$.

Thus $\ker(\varphi) \subseteq \{e_1\}$.

Therefore $\ker(\varphi) = \{e_1\}$.

($\Leftarrow$) Suppose $\ker(\varphi) = \{e_1\}$.

Let $x, y \in G_1$ with $\varphi(x) = \varphi(y)$.

Then $\varphi(x)\varphi(y^{-1}) = \varphi(y)\varphi(y^{-1})$

So, $\varphi(xy^{-1}) = \varphi(yy^{-1})$

Thus, $\varphi(xy^{-1}) = \varphi(e_1)$

So, $\varphi(xy^{-1}) = e_2$.

Thus, $xy^{-1} \in \ker(\varphi)$.

So, $xy^{-1} = e_1$

Then $x = y$

So, φ is one-to-one. $\boxed{///}$

$\boxed{(f)}$

φ is onto iff $\operatorname{im}(\varphi) = G_2$

This is the definition of onto. $\boxed{///}$

7

Suppose $G_1 \cong G_2$ and $H_1 \cong H_2$.

Then, there exist isomorphisms $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : H_1 \to H_2$.

Define $\varphi : G_1 \times H_1 \to G_2 \times H_2$ by

$$\varphi(a,b) = (\varphi_1(a), \varphi_2(b)).$$

Let's show that φ is an isomorphism which will give $G_1 \times H_1 \cong G_2 \times H_2$.

<u>φ is a homomorphism:</u>

Let $(a,b), (c,d) \in G_1 \times H_1$.

Operation in $G_1 \times H_1$

$$\begin{aligned}
\text{Then, } \varphi((a,b)(c,d)) &= \varphi(ac, bd) \\
&= (\varphi_1(ac), \varphi_2(bd)) \\
&= (\varphi_1(a)\varphi_1(c), \varphi_2(b)\varphi_2(d)) \\
&= (\varphi_1(a), \varphi_2(b))(\varphi_1(c), \varphi_2(d)) \\
&= \varphi(a,b)\varphi(c,d).
\end{aligned}$$

φ_1, φ_2 are homomorphisms

Operation in $G_2 \times H_2$

Thus, φ is a homomorphism.

φ is onto: Let $(y_1, y_2) \in G_2 \times H_2$.

Since $y_1 \in G_2$ and $\varphi_1 : G_1 \to G_2$ is onto there exists $x_1 \in G_1$ with $\varphi_1(x_1) = y_1$.

Since $y_2 \in H_2$ and $\varphi_2 : H_1 \to H_2$ is onto there exists $x_2 \in H_1$ with $\varphi_2(x_2) = y_2$.

Then $(x_1, x_2) \in G_1 \times H_1$ and

$$\varphi(x_1, x_2) = (\varphi_1(x_1), \varphi_2(x_2)) = (y_1, y_2).$$

Thus, φ is onto.

φ is one-to-one:

Suppose $\varphi(a,b) = \varphi(c,d)$ where

$$(a,b), (c,d) \in G_1 \times H_1.$$

Then, $(\varphi_1(a), \varphi_2(b)) = (\varphi_1(c), \varphi_2(d))$

So, $\varphi_1(a) = \varphi_1(c)$ and $\varphi_2(b) = \varphi_2(d)$.

Since φ_1 is one-to-one we get $a = c$.

Since φ_2 is one-to-one we get $b = d$.

Thus, $(a,b) = (c,d)$.

So, φ is one-to-one. $\blacksquare$